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BEST CONSTANTS FOR LIPSCHITZ EMBEDDINGS OF METRIC SPACES INTO c_0

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Abstract. We answer a question of Aharoni by showing that every separable metric space can be Lipschitz 2-embedded into c_0 and this result is sharp; this improves earlier estimates of Aharoni, Assouad and Pelant. We use our methods to examine the best constant for Lipschitz embeddings of the classical ℓ_p-spaces into c_0 and give other applications. We prove that if a Banach space embeds almost isometrically into c_0, then it embeds linearly almost isometrically into c_0. We also study Lipschitz embeddings into c_0^+.

1. Introduction

In 1974, Aharoni [1] proved that every separable metric space (M, d) is Lipschitz isomorphic to a subset of the Banach space c_0. Thus, for some constant K, there is a map $f : M \to c_0$ which satisfies the inequality

$$d(x, y) \le \|f(x) - f(y)\| \le K d(x, y) \qquad x, y \in M.$$

Aharoni proved this result with $K = 6 + \epsilon$ where $\epsilon > 0$, so that every separable metric space $(6 + \epsilon)$-embeds into c_0. He also noted that if one takes M to be the Banach space ℓ_1 one cannot have $K < 2$. In fact the map defined by Aharoni took values in the positive cone c_0^+ of c_0. Later Assouad [3] refined Aharoni's result by showing that every separable metric space $(3 + \epsilon)$-embeds into c_0^+ (see [4] p. 176ff). A further improvement was obtained by Pelant in 1994 [10] who showed that every separable metric space 3-embeds into c_0^+ and that this result is sharp in the sense that ℓ_1 cannot be $(3 - \epsilon)$-embedded into c_0^+ (see also [2] for the lower bound).

These results leave open the question of the best constant for Lipschitz embeddings into c_0. Note that c_0 can only be 2-embedded into c_0^+. The main result of this paper is that every separable metric space 2-embeds into c_0 and this is sharp by Aharoni's remark above. To prove this result, for $1 < \lambda \le 2$ we establish a criterion $\Pi(\lambda)$ which is sufficient to imply that a separable metric space λ-embeds into c_0 (and the converse statement is almost true). This criterion enables us to establish sharp results concerning the embedding of ℓ_p-spaces into c_0: thus ℓ_p $2^{1/p}$-embeds into c_0 if $1 \le p < \infty$ and the constant is best possible. Using a previous work of the first author and D. Werner [7], we also show that a Banach space which embeds almost isometrically into c_0 embeds linearly almost isometrically into c_0.

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The same techniques can be applied to embeddings into c_0^+. Here we show that ℓ_p $(2^p+1)^{1/p}$-embeds into c_0^+ and ℓ_p^+ $3^{1/p}$-embeds into c_0^+ and in each case the result is best possible.

We conclude the paper by proving that every separable ultrametric space embeds isometrically into c_0^+ and the infinite branching tree embeds isometrically into c_0.

2. Lipschitz embeddings into c_0

Let (M,d) be a metric space and let A and B be non-empty subsets of M. We define

$$\delta(A,B) = \inf_{a\in A, b\in B} d(a,b)$$

and

$$D(A,B) = \sup_{a\in A, b\in B} d(a,b).$$

In this paper all metric balls are closed with strictly positive radii. If $f:(M_1,d_1)\to(M_2,d_2)$ is a Lipschitz map between metric spaces we write Lip(f) for the Lipschitz constant of f, i.e. the least constant K such that $d_2(f(x),f(y))\le Kd_1(x,y)$ for $x,y\in M_1$.

Lemma 2.1. *Let (M,d) be a metric space and suppose that A,B and C are non-empty subsets of M. Then for $\epsilon>0$, there exists a Lipschitz function $f:M\to\mathbb{R}$ with Lip$(f)\le 1$ such that*
(i) $|f(x)|\le\epsilon \qquad x\in C$
and
(ii) $|f(x)-f(y)| = \theta = \min\big(\delta(A,B),\delta(A,C)+\delta(B,C)+2\epsilon\big) \qquad x\in A,\ y\in B.$

Proof. Let us augment M by adding an extra point 0; let $M^*=M\cup\{0\}$. We define:

$$d^*(x,y) = \begin{cases} \min\big(d(x,y), d(x,C)+d(y,C)+2\epsilon\big) \qquad x,y\in M \\ d(x,C)+\epsilon \qquad x\in M,\ y=0 \\ d(y,C)+\epsilon \qquad x=0,\ y\in M \\ 0 \qquad x=y=0. \end{cases}$$

One can easily check that d^* is a metric on M^*. We can pick s,t in $\mathbb{R}$ such that:

$$-(\delta(B,C)+\epsilon)\le s\le 0\le t\le\delta(A,C)+\epsilon \ \text{ and } \ t-s=\theta.$$

Then we define $g:A\cup B\cup\{0\}\to\mathbb{R}$ by $g=t$ on A, $g=s$ on B and $g(0)=0$. This function is 1-Lipschitz for d^* and can be extended into a 1-Lipschitz function f^* on (M^*,d^*). Let f be the restriction of f^* to M. Then f satisfies the conditions of the Lemma. □

For $\lambda>1$, we say that a metric space (M,d) has property $\Pi(\lambda)$ given any $\mu>\lambda$ there exists $\nu>\mu$ such that if B_1 and B_2 are two metric balls of radii r_1,r_2 respectively then there are finitely many sets $(U_j)_{j=1}^N,(V_j)_{j=1}^N$ such that:

$$\lambda\delta(U_j,V_j)\ge\nu(r_1+r_2) \qquad 1\le j\le N$$

and

$$\{(x,y)\in B_1\times B_2:\ d(x,y)>\mu(r_1+r_2)\}\subset\bigcup_{j=1}^{N}(U_j\times V_j).$$

In this definition the sets U_j, V_j are allowed to be repeated. It is clearly possible, without loss of generality, to assume they are closed. We can also (altering the value of ν) assume that they are open.

Lemma 2.2. *Every metric space has property* $\Pi(2)$.

Proof. For $\mu > 2$, let

$$U = B_1\cap\{x:\ \exists y\in B_2,\ d(x,y)>\mu(r_1+r_2)\}$$

and

$$V = B_2\cap\{y:\ \exists x\in B_1,\ d(x,y)>\mu(r_1+r_2)\}.$$

Then

$$\{(x,y)\in B_1\times B_2:\ d(x,y)>\mu(r_1+r_2)\}\subset U\times V.$$

Suppose $x\in U,\ y\in V$. Let us assume, without loss of generality, that $r_1\le r_2$. Then there exists $x'\in U$ with $d(x',y)>\mu(r_1+r_2)$. Hence

$$d(x,y)>\mu(r_1+r_2)-d(x,x')\ge\mu(r_1+r_2)-2r_1\ge(\mu-1)(r_1+r_2).$$

Therefore we can take $\nu=2\mu-2>\mu$. □

We say that a metric space is *locally compact* (respectively, *locally finite*) if all its metric balls are relatively compact (respectively, finite).

Lemma 2.3. *For any* $\lambda>1$, *every locally compact metric space has property* $\Pi(\lambda)$.

Proof. Let $\mu>\lambda>1$ and B_1, B_2 be two balls of a locally compact metric space (M,d), with respective radii r_1 and r_2. Pick ν such that $\mu<\nu<\lambda\mu$. We denote $\Delta=\{(x,y)\in B_1\times B_2:\ d(x,y)>\mu(r_1+r_2)\}$. Let $\epsilon>0$. Since M is locally compact, there are finitely many points $(x_j,y_j)_{j=1}^N$ in Δ such that

$$\Delta\subset\bigcup_{j=1}^{N}(U_j\times V_j),\quad\text{where}\ \ U_j=B(x_j,\epsilon)\ \text{and}\ V_j=B(y_j,\epsilon).$$

Then, for all $1\le j\le N$, $\lambda\delta(U_j,V_j)>\lambda\mu(r_1+r_2)-2\lambda\epsilon>\nu(r_1+r_2)$, if ϵ was chosen small enough, namely $\epsilon<(2\lambda)^{-1}(\lambda\mu-\nu)(r_1+r_2)$. □

Proposition 2.4. *Let* $\lambda_0\ge1$. *If a metric space* (M,d) λ_0*-embeds into* c_0 *then it has property* $\Pi(\lambda)$ *for every* $\lambda>\lambda_0$.

Proof. Suppose $\mu>\lambda$. Let B_1,B_2 be metric balls of radii r_1,r_2 and centers a_1,a_2. Let $\Delta=\{(x,y)\in B_1\times B_2:\ d(x,y)>\mu(r_1+r_2)\}$. Let $f:M\to c_0$ be an embedding such that

$$d(x,y)\le\|f(x)-f(y)\|\le\lambda_0 d(x,y)\qquad x,y\in M.$$

Suppose $f(x)=(f_i(x))_{i=1}^{\infty}$. Then there exists n so that

$$|f_i(a_1)-f_i(a_2)|<(\mu-\lambda)(r_1+r_2)\qquad i\ge n+1.$$

Thus if $(x,y)\in\Delta$ we have

$$|f_i(x)-f_i(y)|<(\mu-\lambda)(r_1+r_2)+\lambda_0 r_1+\lambda_0 r_2<d(x,y),\qquad i\geq n+1.$$

Hence

$$d(x,y)\leq \max_{1\leq i\leq n}|f_i(x)-f_i(y)|\qquad (x,y)\in\Delta.$$

Choose $\epsilon>0$ so that $\lambda(\mu-\epsilon)>\lambda_0\mu$. By a compactness argument we can find coverings $(W_k)_{k=1}^m$ of B_1 and $(W_k')_{k=1}^{m'}$ of B_2 such that we have

$$|f_i(x)-f_i(x')|\leq \tfrac12\epsilon(r_1+r_2)\qquad x,x'\in W_k,\ 1\leq i\leq n,\ 1\leq k\leq m,$$

and

$$|f_i(x)-f_i(x')|\leq \tfrac12\epsilon(r_1+r_2)\qquad x,x'\in W_k',\ 1\leq i\leq n,\ 1\leq k\leq m'.$$

Let

$$\mathcal{S}=\{(k,k')\ 1\leq k\leq m,\ 1\leq k'\leq m':\ W_k\times W'_{k'}\cap\Delta\neq\emptyset\}$$

and then we define $(U_j)_{j=1}^N,\ (V_j)_{j=1}^N$ in such a way that $(U_j\times V_j)_{j=1}^N$ is an enumeration of $(W_k\times W_{k'})_{(k,k')\in\mathcal{S}}$. Clearly $\Delta\subset\cup_{j=1}^N U_j\times V_j$. Now suppose $x\in U_j,\ y\in V_j$. Then there exist $x'\in U_j,\ y'\in V_j$ so that $d(x',y')>\mu(r_1+r_2)$. Thus there exists $i,\ 1\leq i\leq n$ so that $|f_i(x')-f_i(y')|>\mu(r_1+r_2)$. However

$$|f_i(x)-f_i(y)|\geq |f_i(x')-f_i(y')|-\epsilon(r_1+r_2)>(\mu-\epsilon)(r_1+r_2).$$

Hence

$$\delta(U_j,V_j)\geq\frac{(\mu-\epsilon)}{\lambda_0}(r_1+r_2).$$

Thus we can take $\nu=\lambda\lambda_0^{-1}(\mu-\epsilon)>\mu$. □

We next observe that the definition of $\Pi(\lambda)$ implies a formally stronger conclusion.

Lemma 2.5. *Let (M,d) be a metric space with property $\Pi(\lambda)$. Then for every $\mu>\lambda$ there is a constant $\nu>\mu$ so that if B_1 and B_2 are two metric balls of radii r_1,r_2 respectively then there are finitely many sets $(U_j)_{j=1}^N,(V_j)_{j=1}^N$ such that if $(x,y)\in B_1\times B_2$ and $d(x,y)>\mu(r_1+r_2)$ then there exists $1\leq j\leq N$ so that $x\in U_j,\ y\in V_j$ and:*

$$\lambda\mu\delta(U_j,V_j)\geq\nu d(x,y).$$

Proof. By the definition of $\Pi(\lambda)$ there exists $\nu'>\lambda$ so that such that if B_1 and B_2 are two metric balls of radii r_1,r_2 respectively then there are finitely many sets $(U_j)_{j=1}^N,(V_j)_{j=1}^N$ such that:

$$\lambda\delta(U_j,V_j)\geq\nu'(r_1+r_2)\qquad 1\leq j\leq N$$

and

$$\{(x,y)\in B_1\times B_2:\ d(x,y)>\mu(r_1+r_2)\}\subset\bigcup_{j=1}^N(U_j\times V_j).$$

Suppose $\mu<\nu<\nu'$ and let $\epsilon>0$ be chosen so that $(1+\epsilon)\nu=\nu'$. Let B_1,B_2 be a pair of metric balls of radii $r_1,r_2>0$. Let $D=D(B_1,B_2)$ and let m be the greatest integer such that $(1+\epsilon)^m\mu(r_1+r_2)\leq D$. We define $B_1^{(k)}$ for $0\leq k\leq m$ to be the

ball with the same center as B_1 and radius $(1+\epsilon)^k r_1$; similarly $B_2^{(k)}$ for $0 \le k \le m$ is the ball with the same center as B_2 and radius $(1+\epsilon)^k r_2$. For each $0 \le k \le m$ we may determine sets U_{kl}, V_{kl} for $1 \le l \le N_k$ so that

$$\lambda\delta(U_{kl}, V_{kl}) \ge \nu'(1+\epsilon)^k (r_1 + r_2)$$

and

$$\{(x,y) \in B_1^{(k)} \times B_2^{(k)} :\ d(x,y) > \mu(1+\epsilon)^k (r_1+r_2)\} \subset \bigcup_{l=1}^{N_k} (U_{kl} \times V_{kl}).$$

Now if $x \in B_1, y \in B_2$ with $d(x,y) > \mu(r_1+r_2)$ we may choose $0 \le k \le m$ so that

$$(1+\epsilon)^k \mu(r_1+r_2) < d(x,y) \le (1+\epsilon)^{k+1}\mu(r_1+r_2).$$

Then for a suitable $1 \le l \le N_k$ we have $x \in U_{kl},\ y \in V_{kl}$ and

$$\lambda\mu\delta(U_{kl}, V_{kl}) \ge \nu'(1+\epsilon)^k \mu(r_1+r_2) \ge \frac{\nu'}{1+\epsilon} d(x,y) = \nu d(x,y).$$

Relabeling the sets $(U_{kl}, V_{kl})_{l \le N_k,\ 0 \le k \le m}$ gives the conclusion. □

Lemma 2.6. *Suppose (M,d) has property $\Pi(\lambda)$. Suppose $0 < \alpha < \beta$. Let F, G be finite subsets of M and let $\Delta(F,G,\alpha,\beta)$ be the set of $(x,y) \in M \times M$ such that*

$$\lambda(d(x,G) + d(y,G)) + \alpha \le d(x,y) < \lambda(d(x,F) + d(y,F)) + \beta.$$

Then there is a finite set $\mathcal{F} = \mathcal{F}(F,G,\alpha,\beta)$ of functions $f : M \to \mathbb{R}$ with $Lip(f) \le \lambda$ such that

$$|f(x)| \le \lambda\beta \qquad x \in F$$

and

$$d(x,y) < \max_{f \in \mathcal{F}} |f(x) - f(y)| \qquad (x,y) \in \Delta(F,G,\alpha,\beta).$$

Proof. Let R be the diameter of G. Then for $(x,y) \in \Delta(F,G,\alpha,\beta)$ we have

$$\lambda(d(x,y) - R) + \alpha \le d(x,y)$$

so that

$$(\lambda - 1)d(x,y) < \lambda R.$$

Hence

$$d(x,G) + d(y,G) < \frac{R}{\lambda - 1}.$$

We next let

$$\mu = \lambda + \frac{(\lambda-1)\alpha}{2R}$$

and choose $\nu = \nu(\mu)$ according to the conclusion of Lemma 2.5.
We now fix $\epsilon > 0$ so that $4\mu\epsilon < \alpha$.

Let $E = \{x :\ d(x,G) < (\lambda-1)^{-1}R\}$. Since E is metrically bounded and $F \cup G$ is finite we can partition E into finitely many subsets $(E_1, \dots, E_m)$ so that for each $z \in F \cup G$ we have:

$$|d(x,z) - d(x',z)| \le \epsilon \qquad x, x' \in E_j,\ 1 \le j \le m.$$

Since G is finite, for each j there exist $z_j \in G$ and $r_j \geq 0$ so that

$$\inf_{x \in E_j} d(x, z_j) = \inf_{x \in E_j} d(x, G) = r_j.$$

Thus E_j is contained in a ball B_j centered at z_j with radius $r_j + \epsilon$.
Now for each pair (j,k) we can find finitely many pairs of sets $(U_{jkl}, V_{jkl})_{l=1}^{N_{jk}}$ such that for every $(x,y) \in E_j \times E_k$ with $d(x,y) > \mu(r_j + r_k + 2\epsilon)$ there exists $1 \leq l \leq N_{jk}$ with $x \in U_{jkl}, y \in V_{jkl}$ and

$$\lambda\mu\delta(U_{jkl}, V_{jkl}) \geq \nu d(x,y).$$

We may as well assume that $U_{jkl} \subset E_j$ and $V_{jkl} \subset E_k$.

Then we apply Lemma 2.1 to construct Lipschitz functions $f_{jkl} : M \to \mathbb{R}$ where $1 \leq j,k \leq m,\ 1 \leq l \leq N_{jk}$ such that $\mathrm{Lip}(f_{jkl}) \leq \lambda$,

$$|f_{jkl}(x)| \leq \lambda\beta \qquad x \in F$$

and

$$|f_{jkl}(x) - f_{jkl}(y)| \geq \lambda\theta_{jkl} \qquad x \in U_{jkl},\ y \in V_{jkl}$$

where

$$\theta_{jkl} = \min\big(\delta(U_{jkl}, V_{jkl}), \delta(U_{jkl}, F) + \delta(V_{jkl}, F) + 2\beta\big).$$

Now let us suppose $(x,y) \in \Delta(F, G, \alpha, \beta)$. Then there exists (j,k) so that $x \in E_j,\ y \in E_k$. Note that

$$\begin{aligned}
d(x,y) &\geq \lambda(d(x,G) + d(y,G)) + \alpha \\
&\geq \lambda(r_j + r_k) + \alpha \\
&= \mu(r_j + r_k + 2\epsilon) + \alpha - 2\mu\epsilon - (\mu - \lambda)(r_j + r_k) \\
&\geq \mu(r_j + r_k + 2\epsilon) + \alpha - 2\mu\epsilon - (\mu - \lambda)(\lambda - 1)^{-1}R \\
&> \mu(r_j + r_k + 2\epsilon).
\end{aligned}$$

Thus there exists $1 \leq l \leq N_{jk}$ so that $x \in U_{jkl}, y \in V_{jkl}$ and

$$\lambda\delta(U_{jkl}, V_{jkl}) \geq \frac{\nu}{\mu} d(x,y) > d(x,y).$$

On the other hand, $\epsilon < \alpha/2 < \beta/2$. So

$$\begin{aligned}
\lambda(\delta(U_{jkl}, F) + \delta(V_{jkl}, F) + 2\beta) &\geq \lambda(d(x,F) + d(y,F) + 2\beta - 2\epsilon) \\
&> \lambda(d(x,F) + d(y,F) + \beta) \\
&> d(x,y) + (\lambda - 1)\beta.
\end{aligned}$$

Hence

$$|f_{jkl}(x) - f_{jkl}(y)| \geq \lambda\theta_{jkl} > d(x,y).$$

Thus we can take for $\mathcal{F}$ the collection of all functions f_{jkl} for $1 \leq j,k \leq m$ and $1 \leq l \leq N_{jk}$. $\square$

We now state our main result.

Theorem 2.7. *If a separable metric space (M,d) has property $\Pi(\lambda)$ for $\lambda > 1$, then there is a Lipschitz embedding $f : M \to c_0$ with*

$$d(x,y) < \|f(x) - f(y)\| \leq \lambda d(x,y) \qquad x, y \in M,\ x \neq y.$$

Proof. Let $(u_n)_{n=1}^{\infty}$ be a countable dense set of distinct points of M.
Denote $F_k = \{u_1, \ldots, u_k\}$ for $n \geq 1$. Let $(\epsilon_n)_{n=1}^{\infty}$ be a strictly decreasing sequence with $\lim_{n\to\infty} \epsilon_n = 0$.

Using Lemma 2.6 we can find an increasing sequence of integers $(n_k)_{k=0}^{\infty}$ (with $n_0 = 0$) and a sequence $(f_j)_{j=1}^{\infty}$ of Lipschitz functions $f_j : M \to \mathbb{R}$ with $\mathrm{Lip}(f_j) \leq \lambda$ so that

$$|f_j(x)| \leq \lambda\epsilon_k \qquad x \in F_k,\ n_{k-1} < j \leq n_k$$

and if

$$\lambda(d(x,F_{k+1}) + d(y,F_{k+1})) + \epsilon_{k+1} \leq d(x,y) < \lambda(d(x,F_k) + d(y,F_k)) + \epsilon_k \tag{2.1}$$

then

$$\max_{n_{k-1}<j\leq n_k} |f_j(x) - f_j(y)| > d(x,y).$$

Define the map $f : M \to \ell_\infty$ by $f(x) = (f_j(x))_{j=1}^{\infty}$. Then $\mathrm{Lip}(f) \leq \lambda$ and since f maps each u_j into c_0, $f(M) \subset c_0$. Furthermore if $x \neq y$ the sequence

$$\sigma_k = \lambda(d(x,F_k) + d(y,F_k)) + \epsilon_k$$

is decreasing with $\sigma_1 > d(x,y)$ and $\lim_{k\to\infty} \sigma_k = 0$. Hence there is exactly one choice of k so that (2.1) holds and thus $\|f(x) - f(y)\| > d(x,y)$. $\square$

As a corollary, we obtain the following improvement of Aharoni's theorem.

Theorem 2.8. *For every separable metric space (M,d) there is a Lipschitz embedding $f : M \to c_0$ so that*

$$d(x,y) < \|f(x) - f(y)\| \leq 2d(x,y) \qquad x, y \in M,\ x \neq y.$$

Proof. Combine Lemma 2.2 and Theorem 2.7. $\square$

Remark. It follows from Proposition 3 in Aharoni's original paper [1] that the above statement is optimal.

Theorem 2.9. *For every locally compact metric space (M,d) and every $\lambda > 1$, (M,d) λ-embeds into c_0. This result is best possible.*

Proof. The existence of the embedding follows immediately from the combination of Lemma 2.3 and Theorem 2.7. The optimality of the statement follows from Proposition 3.2 in [10], where J. Pelant proved that $[0,1]^{\mathbb{N}}$ equipped with the distance $d((x_n),(y_n)) = \sum 2^{-n}|x_n - y_n|$ cannot be isometrically embedded into c_0. To complete the the picture we shall now give a locally finite counterexample.

Let $(e_n)_{n=0}^{\infty}$ be the canonical basis of ℓ_1 and consider the following locally finite metric subspace of ℓ_1: $M = \{0, e_0\} \cup \{ne_n, e_0 + ne_n;\ n \geq 1\}$. Assume that $f = (f_k)_{k=1}^{\infty}$ is an isometry from M into c_0 such that $f(0) = 0$. Then for all $n \neq m$ in $\mathbb{N}$, there exists $k = k_{n,m} \geq 1$ such that $|f_k(e_0 + ne_n) - f_k(me_m)| = 1 + n + m$. Since

$f_k(0) = 0$, we obtain that there is $\varepsilon = \varepsilon_{n,m} \in \{-1, 1\}$ such that $f_k(e_0 + ne_n) = \varepsilon(1+n)$ and $f_k(me_m) = -\varepsilon m$. Therefore $f_k(e_0) = \varepsilon$ and $f_k(ne_n) = \varepsilon n$.
Since $f(e_0) \in c_0$, there exists an integer K such that for all positive integers $n \neq m$, $k_{n,m} \leq K$. Hence, if $\alpha(k,n)$ is the signum of $f_k(ne_n)$, we have that there exists $k \leq K$ so that $\alpha(k,n) \neq \alpha(k,m)$, whenever $1 \leq n < m$. But on the other hand, there is clearly an infinite subset A of $\mathbb{N}$ such that for every $k \leq K$ and every $n, m \in A$, $\alpha(k,n) = \alpha(k,m)$. This is a contradiction. $\square$

3. Embeddings of classical Banach spaces

In this section we will consider the best constants for embedding certain classical spaces into c_0. We start by establishing a lower bound condition, using the Borsuk-Ulam theorem.

Proposition 3.1. *Suppose X is a Banach space and that $f : X \to c_0$ is a Lipschitz embedding with constant λ_0. Then for any $u \in X$ with $\|u\| = 1$ and any infinite-dimensional subspace Y of X we have*

$$\inf_{\substack{y\in Y\\ \|y\|=1}} \|u + y\| \leq \lambda_0.$$

Proof. It follows from Lemma 2.4 that X has property $\Pi(\lambda)$ for any $\lambda > \lambda_0$. Let us consider $B_1 = -u + B_X$ and $B_2 = u + B_X$, where B_X denotes the closed unit ball of X. Suppose $\mu > \lambda_0$ and select $\mu > \lambda > \lambda_0$. Then, for some $\nu > \mu$, we can find finitely many sets $(U_j, V_j)_{j=1}^N$ (which we can assume to be closed) verifying:

$$\lambda\delta(U_j, V_j) \geq 2\nu$$

and

$$\{(x,y) \in B_1 \times B_2 : \ \|x - y\| > 2\mu\} \subset \cup_{j=1}^N U_j \times V_j.$$

Now let E be any subspace of X of dimension greater than N and let

$$A_j = \{e \in E : \ \|e\| = 1, \ (-u + e, u - e) \in U_j \times V_j\}.$$

Thus the sets A_j are all closed subsets of the unit sphere S_E of E. Assume that for any $e \in S_E$, $\|u - e\| > \mu$. Then $A_1 \cup \cdots \cup A_N = S_E$. We now use a classical corollary of the Borsuk-Ulam theorem which is in fact due to Lyusternik and Shnirelman [8] and predates Borsuk's work (see [9] p. 23). This gives the existence of e in S_E and $k \leq N$ such that e and $-e$ belong to A_k, i.e. $-u \pm e \in U_k$ and $u \pm e \in V_k$. This implies that $\delta(U_k, V_k) \leq 2$ and hence $\lambda \geq \nu > \mu$ which is a contradiction. Thus there exists $e \in S_E$ with $\|u - e\| \leq \mu$.
Since this is true for every finite-dimensional subspace E of dimension greater than N and every $\mu > \lambda_0$ the conclusion follows. $\square$

Theorem 3.2. *Suppose $1 \leq p < \infty$. Then there is a Lipschitz embedding of ℓ_p into c_0 with constant $2^{1/p}$, and this constant is best possible.*

Proof. The fact that ℓ_p does not λ-embed into c_0 when $\lambda < 2^{1/p}$ follows immediately from Proposition 3.1. So we only need to show that ℓ_p verifies condition $\Pi(2^{1/p})$.

Let B_1 and B_2 be balls with centers a_1, a_2 and radii r_1, r_2. Suppose $\mu > 2^{1/p}$. Then $\mu < 2^{1/p}(\mu^p - 1)^{1/p}$. We pick ν such that $\mu < \nu < 2^{1/p}(\mu^p - 1)^{1/p}$ and we fix $\epsilon > 0$ so that

$$2^{1/p}\big(\mu^p(r_1+r_2)^p - (r_1+r_2+2\epsilon)^p\big)^{1/p} - 2^{1+1/p}\epsilon > \nu(r_1+r_2).$$

We first select $N \in \mathbb{N}$ so that

$$\sum_{k=N+1}^{\infty} |a_1(k)|^p, \sum_{k=N+1}^{\infty} |a_2(k)|^p < \epsilon^p.$$

Let E be the linear span of $\{e_1, \dots, e_N\}$ where $(e_j)_{j=1}^{\infty}$ is the canonical basis of ℓ_p. Let P the canonical projection of ℓ_p onto E, $Q = I - P$ and $R = \max(\|a_1\| + r_1, \|a_2\| + r_2)$. Then we partition RB_E into finitely many sets $A_1, \dots, A_m$ with diam $A_j < \epsilon$. Now, set $U_j = \{x \in B_1 : \ Px \in A_j\}$, $V_j = \{x \in B_2 : \ Px \in A_j\}$ and

$$\mathcal{S} = \{(j,k) \ \exists (x,y) \in U_j \times V_k : \ \|x - y\| > \mu(r_1+r_2)\}.$$

Thus we have

$$\{(x,y) \in B_1 \times B_2 : \ \|x-y\| > \mu(r_1+r_2)\} \subset \bigcup_{(j,k)\in\mathcal{S}} U_j \times V_k.$$

It remains to estimate $\delta(U_j, V_k)$ for $(j,k) \in \mathcal{S}$. Suppose $u \in U_j, v \in V_k$ and that $x \in U_j, y \in V_k$ are such that $\|x - y\| > \mu(r_1 + r_2)$. Then

$$\|u - v\| \geq \|Pu - Pv\| \geq \|Px - Py\| - 2\epsilon.$$

On the other hand

$$r_1 \geq \|x - a_1\| \geq \|Qx - Qa_1\| \geq \|Qx\| - \epsilon$$

and

$$r_2 \geq \|y - a_2\| \geq \|Qy - Qa_2\| \geq \|Qy\| - \epsilon.$$

Thus

$$\|Qx - Qy\| \leq r_1 + r_2 + 2\epsilon.$$

Now

$$\mu^p(r_1+r_2)^p < \|Px - Py\|^p + \|Qx - Qy\|^p \leq \|Px - Py\|^p + (r_1+r_2+2\epsilon)^p.$$

Hence

$$\|Px - Py\|^p > \mu^p(r_1+r_2)^p - (r_1+r_2+2\epsilon)^p$$

and thus

$$2^{1/p}\delta(U_j, V_k) \geq 2^{1/p}\big(\mu^p(r_1+r_2)^p - (r_1+r_2+2\epsilon)^p\big)^{1/p} - 2^{1+1/p}\epsilon > \nu(r_1+r_2).$$

□

We now give a second lower bound condition in place of Proposition 3.1. We do not know whether the conclusion can be improved replacing λ_0^3 by λ_0. If X has a 1-unconditional basis, λ_0^3 can be improved to λ_0^2.

Proposition 3.3. *If X is a separable Banach space and $f : X \to c_0$ is a Lipschitz embedding with constant λ_0 then if $\|x\| = 1$ and $(x_n)_{n=1}^{\infty}$ is a normalized weakly null sequence in X we have:*

$$\limsup_{n\to\infty} \|x + x_n\| \le \lambda_0^3. \tag{3.2}$$

Proof. We assume that $\|x-y\| \le \|f(x)-f(y)\| \le \lambda_0\|x-y\|$ for $x, y \in X$. Let $\mathcal{U}$ be a non-principal ultrafilter on the natural numbers $\mathbb{N}$. We start by proving that if $x \in X$ and $(y_n)_{n=1}^{\infty}, (z_n)_{n=1}^{\infty}$ are two weakly null sequences with $\lim_{n\in\mathcal{U}} \|y_n\|, \lim_{n\in\mathcal{U}} \|z_n\| \le \|x\|$ then

$$\lambda_0^{-1} \lim_{n\in\mathcal{U}} \|2x + y_n + z_n\| \le \lim_{m\in\mathcal{U}} \lim_{n\in\mathcal{U}} \|2x + y_m + z_n\| \le \lambda_0 \lim_{n\in\mathcal{U}} \|2x + y_n + z_n\|. \tag{3.3}$$

Indeed it suffices to show this under the condition $\lim_{n\in\mathcal{U}} \|y_n\| = \alpha, \lim_{n\in\mathcal{U}} \|z_n\| = \beta$ where $\alpha, \beta \le 1$ and $\|x\| = 1$. Fix any $\epsilon > 0$. Let $f(x) = (f_j(x))_{j=1}^{\infty}$. Then for some N we have

$$|f_j(x) - f_j(-x)| < \epsilon \qquad j > N.$$

Thus

$$|f_j(x + y_m) - f_j(-x - z_n)| \le \lambda_0(\|y_m\| + \|z_n\|) + \epsilon \qquad j > N.$$

Hence

$$\lim_{m\in\mathcal{U}} \lim_{n\in\mathcal{U}} \max_{j>N} |f_j(x + y_m) - f_j(-x - z_n)| \le \lambda_0(\alpha + \beta) + \epsilon.$$

and

$$\lim_{n\in\mathcal{U}} \max_{j>N} |f_j(x + y_n) - f_j(-x - z_n)| \le \lambda_0(\alpha + \beta) + \epsilon.$$

Let $\sigma_j = \lim_{n\in\mathcal{U}} f_j(x + y_n)$ and $\tau_j = \lim_{n\in\mathcal{U}} f_j(-x - z_n)$. Then

$$\lim_{n\in\mathcal{U}} |f_j(x + y_n) - f_j(-x - z_n)| = |\sigma_j - \tau_j|$$

and

$$\lim_{m\in\mathcal{U}} \lim_{n\in\mathcal{U}} |f_j(x + y_m) - f_j(-x - z_n)| = |\sigma_j - \tau_j|.$$

Thus

$$\begin{aligned}
\lim_{n\in\mathcal{U}} \|2x + y_n + z_n\| &\le \lim_{n\in\mathcal{U}} \|f(x + y_n) - f(-x - z_n)\| \\
&\le \max(\max_{1\le j\le N} |\sigma_j - \tau_j|, \lambda_0(\alpha + \beta) + \epsilon) \\
&\le \max(\lambda_0 \lim_{m\in\mathcal{U}} \lim_{n\in\mathcal{U}} \|2x + y_m + z_n\|, \lambda_0(\alpha + \beta) + \epsilon).
\end{aligned}$$

Noting that $\epsilon > 0$ is arbitrary and that

$$\alpha + \beta \le 2 \le \lim_{m\in\mathcal{U}} \lim_{n\in\mathcal{U}} \|2x + y_m + z_n\|$$

we obtain that

$$\lim_{n\in\mathcal{U}} \|2x + y_n + z_n\| \le \lambda_0 \lim_{m\in\mathcal{U}} \lim_{n\in\mathcal{U}} \|2x + y_m + z_n\|.$$

The other inequality in (3.3) is similar.

Now choose $x_n = y_n = -z_n$ in (3.3). We obtain

$$\lim_{m\in\mathcal{U}} \lim_{n\in\mathcal{U}} \|2x + x_m - x_n\| \le 2\lambda_0\|x\|$$

provided $(x_n)_{n=1}^\infty$ is weakly null and $\lim_{n\in\mathcal{U}}\|x_n\|\le\|x\|$. Hence

$$\lim_{m\in\mathcal{U}}\|x+\tfrac{1}{2}x_m\|\le\lambda_0\|x\|.$$

This inequality can be iterated to show that

$$\lim_{m\in\mathcal{U}}\lim_{n\in\mathcal{U}}\|x+\tfrac{1}{2}x_m+\tfrac{1}{2}x_n\|\le\lambda_0^2\|x\|.$$

Now assume $\|x\|=1$ and $(x_n)_{n=1}^\infty$ is a normalized weakly null sequence. Then

$$\begin{aligned}\lim_{n\in\mathcal{U}}\|x+x_n\| &= \frac{1}{2}\lim_{n\in\mathcal{U}}\|2x+x_n+x_n\|\\ &\le \frac{1}{2}\lambda_0\lim_{m\in\mathcal{U}}\lim_{n\in\mathcal{U}}\|2x+x_m+x_n\|\\ &\le \lambda_0^3.\end{aligned}$$

□

Theorem 3.4. *Let X be a separable Banach space.*
(i) If X isometrically embeds into c_0, then X is linearly isometric to a closed subspace of c_0.
(ii) If, for every $\epsilon>0$, X Lipschitz embeds into c_0 with constant at most $1+\epsilon$, then, for every $\epsilon>0$ there is a closed subspace Y_ϵ of c_0 with Banach-Mazur distance $d(X,Y_\epsilon)<1+\epsilon$.

Proof. (i) is a direct consequence of the result of [5] that if a *separable* Banach space is isometric to a subset of a Banach space Z then it is also linearly isometric to a subspace of Z.

(ii) Here we observe first that if X contains a subspace isomorphic to ℓ_1 then, for any $\epsilon>0$, it contains a subspace Z_ϵ with the Banach-Mazur distance $d(Z_\epsilon,\ell_1)\le 1+\epsilon$ by James' distortion theorem [6]. Assume now that X can be $\lambda-$embedded into c_0. Thus we have that for any $\epsilon>0$, ℓ_1 can be $\lambda(1+\epsilon)-$embedded into c_0. Then it follows from Aharoni's counterexample in [1] that $\lambda\ge 2$.

Suppose now that X does not contain any isomorphic copy of ℓ_1. If $\|x\|=1$ and $(x_n)_{n=1}^\infty$ is any normalized weakly null sequence we have by Proposition 3.3 that

$$\lim_{n\to\infty}\|x+x_n\|=1.$$

The conclusion then follows from [7] Theorem 3.5. □

4. Embeddings into c_0^+.

In this section and the following we complete the already thorough study of Lipschitz embeddings into c_0^+ made by Pelant in [10].

Lemma 4.1. *Let (M,d) be a metric space and suppose that A,B and C are non-empty subsets of M. Then for $\epsilon>0$, there exists a Lipschitz function $f:M\to\mathbb{R}_+$ with $Lip(f)\le 1$ such that*
(i) $f(x)\le\epsilon \qquad x\in C$
and
(ii) $|f(x)-f(y)|\ge\theta=\min\big(\delta(A,B),\max(\delta(A,C),\delta(B,C))+\epsilon\big) \qquad x\in A,\ y\in B.$

Proof. Let us suppose $\delta(A,C) \geq \delta(B,C)$. Thus $\theta = \min(\delta(A,B), \delta(A,C) + \epsilon)$. Let us define:

$$f(x) = \max(\theta - d(x,A), 0) \qquad x \in M.$$

Then $f(x) = \theta$ for $x \in A$. If $x \in B$ $f(x) \leq \theta - \delta(A,B)$ so that $f(x) = 0$, while if $x \in C$ we have

$$f(x) \leq \theta - \delta(A,C) \leq \epsilon.$$

□

We may now introduce a condition analogous to $\Pi(\lambda)$. We say that (M,d) has property $\Pi_+(\lambda)$, where $\lambda > 1$, if:
(i) Whenever $\mu > \lambda$ there exists $\nu > \mu$ so that if B_1 and B_2 are two metric balls of the same radius r, there is a finite number of sets $(U_j)_{j=1}^N$ and $(V_j)_{j=1}^N$ so that

$$\lambda\delta(U_j, V_j) \geq \nu r$$

and

$$\{(x,y) \in B_1 \times B_2 : \ d(x,y) > \mu r\} \subset \bigcup_{j=1}^N U_j \times V_j,$$

and
(ii) If $1 < \lambda \leq 2$, there exists $1 < \theta < \lambda$ and a function $\varphi : M \to [0,\infty)$ so that

$$|\varphi(x) - \varphi(y)| \leq d(x,y) \leq \theta \max(\varphi(x), \varphi(y)) \qquad x, y \in M. \tag{4.4}$$

Let us note here that condition (ii) is not required when $\lambda > 2$ since fixing any $a \in X$ the function $\varphi(x) = d(x,a)$ satisfies (4.4) with $\theta = 2$.
We can repeat the same program for property $\Pi_+(\lambda)$.

Lemma 4.2. *Every metric space has property* $\Pi_+(3)$.

Proof. For $\mu > 3$, let

$$U = B_1 \cap \{x : \ \exists y \in B_2, \ d(x,y) > \mu r\}$$

and

$$V = B_2 \cap \{y : \ \exists x \in B_1, \ d(x,y) > \mu r\}.$$

Then

$$\{(x,y) \in B_1 \times B_2 : \ d(x,y) > \mu r\} \subset U \times V.$$

Suppose $x \in U, \ y \in V$. Then there exists $x' \in U$ with $d(x',y) > \mu r$. Hence

$$d(x,y) > \mu r - d(x,x') \geq (\mu - 2)r.$$

Therefore we can take $\nu = 3\mu - 6 > \mu$. □

Lemma 4.3. *For any $\lambda > 2$, every locally compact metric space has property* $\Pi_+(\lambda)$.

The proof is immediate. Let us mention that a locally compact metric space satisfies condition (i) for every $\lambda > 1$.

We also have

Lemma 4.4. *For any $\lambda > 1$, every compact metric space has property* $\Pi_+(\lambda)$.

Proof. Let (K,d) be a compact metric space. We only have to prove condition (ii). For $\epsilon > 0$, pick a finite ϵ-net F of K and define $\varphi_\epsilon(x) = \max(d(x,z))_{z\in F}$. For a given $\lambda > 1$, φ_ϵ fulfills condition (ii) of $\Pi_+(\lambda)$ if ϵ is small enough. □

Proposition 4.5. *Suppose $\lambda_0 \geq 1$ and M is a metric space which Lipschitz embeds into c_0^+ with constant λ_0. Then M has property $\Pi_+(\lambda)$ for all $\lambda > \lambda_0$.*

Proof. We first consider (i) of the definition of $\Pi_+(\lambda)$. Suppose $\mu > \lambda > \lambda_0$. Let B_1, B_2 be metric balls of radii $r > 0$ and centers a_1, a_2. Let $\Delta = \{(x,y) \in B_1 \times B_2 : d(x,y) > \mu r\}$. Let $f : M \to c_0^+$ be an embedding such that

$$d(x,y) \leq \|f(x) - f(y)\| \leq \lambda_0 d(x,y) \qquad x,y \in M.$$

Suppose $f(x) = (f_i(x))_{i=1}^\infty$. Then there exists n so that

$$f_i(a_1), f_i(a_2) < (\mu - \lambda)r \qquad i \geq n+1.$$

Thus if $(x,y) \in \Delta$ we have

$$|f_i(x) - f_i(y)| \leq \max(f_i(x), f_i(y)) < (\mu-\lambda)r + \lambda_0 r < d(x,y), \qquad i \geq n+1.$$

Hence

$$d(x,y) \leq \max_{1\leq i\leq n} |f_i(x) - f_i(y)| \qquad (x,y) \in \Delta.$$

Choose $\epsilon > 0$ so that $\lambda(\mu - \epsilon) > \lambda_0 \mu$. By a compactness argument we can find coverings $(W_k)_{k=1}^m$ of B_1 and $(W_k')_{k=1}^{m'}$ of B_2 such that:

$$|f_i(x) - f_i(x')| \leq \tfrac{1}{2}\epsilon r \qquad x, x' \in W_k,\ 1 \leq i \leq n,\ 1 \leq k \leq m,$$

and

$$|f_i(x) - f_i(x')| \leq \tfrac{1}{2}\epsilon r \qquad x, x' \in W_k',\ 1 \leq i \leq n,\ 1 \leq k \leq m'.$$

Let

$$\mathcal{S} = \{(k,k')\ 1 \leq k \leq m,\ 1 \leq k' \leq m' :\ W_k \times W_{k'}' \cap \Delta \neq \emptyset\}$$

and define $(U_j)_{j=1}^N,\ (V_j)_{j=1}^N$ in such a way that $(U_j \times V_j)_{j=1}^N$ is an enumeration of $(W_k \times W_{k'})_{(k,k')\in\mathcal{S}}$. Then $\Delta \subset \cup_{j=1}^N U_j \times V_j$ and the same calculations as in the proof of Proposition 2.4 give that

$$\lambda\delta(U_j, V_j) \geq \nu r \quad \text{with} \quad \nu = \lambda\lambda_0^{-1}(\mu - \epsilon) > \mu.$$

If $\lambda \leq 2$ we also must consider (ii). Here we define $\varphi(x) = \lambda_0^{-1}\|f(x)\|$ where $f : M \to c_0^+$ is as above. Then φ satisfies (4.4) with $\theta = \lambda_0$. Indeed,

$$|\varphi(x) - \varphi(y)| \leq \lambda_0^{-1}\|f(x) - f(y)\| \leq d(x,y)$$

and

$$d(x,y) \leq \|f(x) - f(y)\| \leq \max(\|f(x)\|, \|f(y)\|) \leq \lambda_0 \max(\varphi(x), \varphi(y)).$$

□

Next, in place of Lemma 2.5 we have

Lemma 4.6. *Let $\lambda > 1$ and (M, d) be a metric space with property $\Pi_+(\lambda)$. Then for every $\mu > \lambda$ there is a constant $\nu > \mu$ so that if B_1 and B_2 are two metric balls of radius r then there are finitely many sets $(U_j)_{j=1}^N, (V_j)_{j=1}^N$ such that if $(x, y) \in B_1 \times B_2$ and $d(x, y) > \mu r$ then there exists $1 \le j \le N$ so that $x \in U_j,\ y \in V_j$ and:*

$$\lambda\mu\delta(U_j, V_j) \ge \nu d(x, y).$$

We omit the proof of this which is very similar to that of Lemma 2.5 and only uses part (i) of the definition of $\Pi_+(\lambda)$.

Then we have the following analogue of Lemma 2.6.

Lemma 4.7. *Let $\lambda > 1$. Suppose (M, d) has property $\Pi_+(\lambda)$. Suppose $0 < \alpha < \beta$. Let F, G be finite subsets of M and let $\Delta_+(F, G, \alpha, \beta)$ be the set of $(x, y) \in M \times M$ such that*

$$\lambda \max(d(x, G), d(y, G)) + \alpha \le d(x, y) < \lambda \max(d(x, F), d(y, F)) + \beta.$$

Then there is a finite set $\mathcal{F} = \mathcal{F}(F, G, \alpha, \beta)$ of functions $f : M \to \mathbb{R}_+$ with Lip$(f) \le \lambda$ and such that

$$f(x) \le \lambda\beta \qquad x \in F$$

and

$$d(x, y) < \max_{f\in\mathcal{F}} |f(x) - f(y)| \qquad (x, y) \in \Delta_+(F, G, \alpha, \beta).$$

Proof. We first argue that for some constant K we have

$$d(x, y) \le K, \qquad x, y \in \Delta_+(F, G, \alpha, \beta).$$

If $\lambda > 2$ this is follows from the fact that

$$d(x, G) + d(y, G) \ge d(x, y) - R$$

where R is the diameter of G. Hence

$$d(x, y) \le K = \lambda(\lambda - 2)^{-1}R, \qquad x, y \in \Delta_+(F, G, \alpha, \beta).$$

In the case $1 < \lambda \le 2$ let φ, θ be as in the definition of $\Pi_+(\lambda)$ and satisfy (4.4). Let $K_0 = \max\{\varphi(z) :\ z \in G\}$. Thus

$$\begin{aligned}\lambda d(x, y) &\le \lambda\theta \max(\varphi(x), \varphi(y))\\ &\le \lambda\theta K_0 + \lambda\theta \max(d(x, G), d(y, G))\\ &\le \lambda\theta K_0 + \theta d(x, y) \qquad x, y \in \Delta_+(F, G, \alpha, \beta),\end{aligned}$$

so that

$$d(x, y) \le K = \frac{\lambda\theta K_0}{\lambda - \theta}, \qquad x, y \in \Delta_+(F, G, \alpha, \beta).$$

We next let

$$\mu = \lambda + \frac{\alpha\lambda}{2K}$$

and choose $\nu = \nu(\mu)$ according to the conclusion of Lemma 4.6. We fix $\epsilon > 0$ so that $\epsilon < \min(\frac{\alpha}{2}, \lambda^{-1}(\lambda - 1)\beta)$.

Let $E = \{x : \ d(x,G) \le \lambda^{-1}K\}$. Since E is metrically bounded and $F \cup G$ is finite we can partition E into finitely many subsets $(E_1, \dots, E_m)$ so that for each $z \in F \cup G$ we have:

$$|d(x,z) - d(x',z)| \le \epsilon \qquad x, x' \in E_j, \ 1 \le j \le m.$$

For each j, we define $z_j \in G$ and r_j, as in the proof of Lemma 2.6, so that

$$\inf_{x \in E_j} d(x, z_j) = \inf_{x \in E_j} d(x, G) = r_j.$$

Note that $r_j \le \lambda^{-1}K$ and E_j is contained in a ball B_j centered at z_j with radius $r_j + \epsilon$.
Now for each pair (j,k) we denote $B_{j,k}$ the ball with center z_j and radius $\max(r_j + \epsilon, r_k + \epsilon)$. By Lemma 4.6, we can find finitely many pairs of sets $(\tilde{U}_{jkl}, \tilde{V}_{jkl})_{l=1}^{N_{jk}}$ such that for every $(x,y) \in B_{j,k} \times B_{k,j}$ with $d(x,y) > \mu(\max(r_j, r_k) + \epsilon)$ there exists $1 \le l \le N_{jk}$ with $x \in \tilde{U}_{jkl}, y \in \tilde{V}_{jkl}$ and

$$\lambda\mu\delta(\tilde{U}_{jkl}, \tilde{V}_{jkl}) \ge \nu d(x,y).$$

Then we set $U_{jkl} = \tilde{U}_{jkl} \cap E_j$ and $V_{jkl} = \tilde{V}_{jkl} \cap E_k$.

We now apply Lemma 4.1 to construct Lipschitz functions $f_{jkl} : M \to \mathbb{R}_+$ where $1 \le j, k \le m, \ 1 \le l \le N_{jk}$ such that $\mathrm{Lip}(f_{jkl}) \le \lambda$,

$$f_{jkl}(x) \le \lambda\beta \qquad x \in F$$

and

$$|f_{jkl}(x) - f_{jkl}(y)| \ge \lambda\theta_{jkl} \qquad x \in U_{jkl}, \ y \in V_{jkl}$$

where

$$\theta_{jkl} = \min\big(\delta(U_{jkl}, V_{jkl}), \max(\delta(U_{jkl}, F), \delta(V_{jkl}, F)) + \beta\big).$$

Now let us suppose $(x,y) \in \Delta_+(F, G, \alpha, \beta)$. Then there exists (j,k) so that $x \in E_j, \ y \in E_k$. It follows from our choice of μ and ϵ that

$$\begin{aligned} d(x,y) &\ge \lambda \max(d(x,G), d(y,G)) + \alpha \\ &\ge \lambda \max(r_j, r_k) + \alpha > \mu \max(r_j, r_k) + \epsilon. \end{aligned}$$

Thus there exists $1 \le l \le N_{jk}$ so that $x \in U_{jkl}, y \in V_{jkl}$ and

$$\lambda\delta(U_{jkl}, V_{jkl}) \ge \frac{\nu}{\mu} d(x,y) > d(x,y).$$

On the other hand, $\epsilon < \lambda^{-1}(\lambda - 1)\beta$, so

$$\begin{aligned} \lambda \max(\delta(U_{jkl}, F), \delta(V_{jkl}, F)) + \beta) &\ge \lambda \max(d(x,F), d(y,F)) + \lambda(\beta - \epsilon) \\ &> \lambda \max(d(x,F), d(y,F)) + \beta \\ &> d(x,y). \end{aligned}$$

Hence

$$|f_{jkl}(x) - f_{jkl}(y)| \ge \lambda\theta_{jkl} > d(x,y).$$

Thus we can take for $\mathcal{F}$ the collection of all functions f_{jkl} for $1 \le j, k \le m, \ 1 \le l \le N_{jk}$. $\square$

Finally our theorem is

Theorem 4.8. *Suppose a separable metric space (M,d) has property $\Pi_+(\lambda)$. Then there is a Lipschitz embedding $f : M \to c_0^+$ with*

$$d(x,y) < \|f(x) - f(y)\| \le \lambda d(x,y) \qquad x, y \in M,\ x \neq y.$$

Proof. We use the notation of the proof of Theorem 2.7. Then we build an increasing sequence of integers $(n_k)_{k=0}^\infty$ (with $n_0 = 0$) and a sequence $(f_j)_{j=1}^\infty$ of Lipschitz functions $f_j : M \to \mathbb{R}_+$ with $\mathrm{Lip}(f_j) \le \lambda$ so that

$$f_j(x) \le \lambda\epsilon_k \qquad x \in F_k,\ n_{k-1} < j \le n_k$$

and if

$$\lambda \max(d(x,F_{k+1}), d(y,F_{k+1})) + \epsilon_{k+1} \le d(x,y) < \lambda \max(d(x,F_k), d(y,F_k)) + \epsilon_k \tag{4.5}$$

then

$$\max_{n_{k-1} < j \le n_k} |f_j(x) - f_j(y)| > d(x,y).$$

If $x \neq y$ the sequence

$$\tau_k = \lambda \max(d(x,F_k), d(y,F_k)) + \epsilon_k$$

is decreasing and tends to zero.
If $\lambda > 2$, we clearly have $\tau_1 > d(x,y)$.
Assume $1 < \lambda \le 2$. Let φ be given by the part (ii) of property $\Pi_+(\lambda)$. We choose $\epsilon_1 > \lambda\varphi(u_1)$. Then we have

$$d(x,y) \le \lambda \max(\varphi(x), \varphi(y)) < \epsilon_1 + \lambda \max(d(x,u_1), d(y,u_1)) = \tau_1.$$

Hence, in both cases the desired embedding can be defined again by $f(x) = (f_j(x))_{j=1}^\infty$. $\square$

As a first corollary, we obtain the two following results due to Pelant ([10]).

Corollary 4.9. *(a) For every separable metric space (M,d) there is a Lipschitz embedding $f : M \to c_0^+$ so that*

$$d(x,y) < \|f(x) - f(y)\| \le 3d(x,y) \qquad x, y \in M,\ x \neq y.$$

(b) For any compact metric space (K,d) and any $\lambda > 1$, (K,d) λ-embeds into c_0^+.

It is proved in [10] that both of the above statements are optimal. This was also known to Aharoni [2] for part (a).

We also have.

Theorem 4.10. *For every locally compact metric space (M,d) and every $\lambda > 2$, (M,d) λ-embeds into c_0^+. This result is optimal.*

Proof. The result is obtained by combining Theorem 4.8 and Lemma 4.3. We only have to show its optimality.

Let $\mathcal{D}$ be the set of all finite sequences with values in $\{0,1\}$ including the empty sequence denoted $\emptyset$ and let $\mathcal{D}^* = \mathcal{D} \setminus \{\emptyset\}$. For $s \in \mathcal{D}$, we denote $|s|$ its length. Then

$(e_s)_{s\in\mathcal{D}}$ is the canonical basis of $\ell_1(\mathcal{D})$. We consider the following metric subspace of $\ell_1(\mathcal{D})$:

$$M = \{0, e_\emptyset\} \cup \{|s|e_s, e_\emptyset + |s|e_s,\ s \in \mathcal{D}^*\}.$$

This is clearly a locally finite metric space. Assume now that there exists $f = (f_k)_{k=1}^\infty : M \to c_0^+$ such that

$$\|x-y\|_1 \le \|f(x)-f(y)\|_\infty \le 2\|x-y\|_1 \qquad x, y \in M.$$

There exits $K \ge 1$ such that $f_k(e_\emptyset) < 1$ and $f_k(0) < 1$ for all $k > K$. Then, using the positivity of f, we obtain

$$\begin{aligned}|f_k(e_\emptyset + ne_s) - f_k(ne_t)| &\le \max\big(f_k(e_\emptyset + ne_s), f_k(ne_t)\big)\\ &< 1+2n \qquad k > K,\ s \ne t,\ |s| = |t| = n.\end{aligned}$$

On the other hand,

$$\|f(e_\emptyset + ne_s) - f(ne_t)\|_\infty \ge 1+2n \qquad s \ne t,\ |s| = |t| = n.$$

Thus, for all $s \ne t,\ |s| = |t| = n$, there exists $k \le K$ so that

$$|f_k(e_\emptyset + ne_s) - f_k(ne_t)| \ge 1+2n.$$

Let now $C = \max(\|f(e_\emptyset)\|_\infty, \|f(0)\|_\infty)$. Then

$$|f_k(e_\emptyset + ne_s) - f_k(ne_t)| \le C+2n \qquad k \le K,\ s \ne t,\ |s| = |t| = n.$$

Thus, for n large enough and all $s \ne t,\ |s| = |t| = n$, there exists $k \le K$ such that either

$$f_k(ne_s) \le C-1 \qquad \text{and} \qquad f_k(e_\emptyset + ne_t) \ge 1+2n$$

or

$$f_k(ne_s) \ge 1+2n \qquad \text{and} \qquad f_k(e_\emptyset + ne_t) \le C-1.$$

Therefore: either

$$f_k(ne_s) \le C-1 \qquad \text{and} \qquad f_k(ne_t) \ge 2n-1$$

or

$$f_k(ne_s) \ge 1+2n \qquad \text{and} \qquad f_k(ne_t) \le C+1.$$

Let us now denote $\alpha(k,s) = \mathbb{1}_{[0,C+1]}(f_k(|s|e_s))$. Then, for n big enough, we have that for all $s \ne t,\ |s| = |t| = n$, there exists $k \le K$ so that $\alpha(k,s) \ne \alpha(k,t)$. This is clearly impossible if $n > K$. This finishes our proof.

$\square$

5. Embeddings of subsets of classical Banach spaces into c_0^+.

Proposition 5.1. *Suppose X is a separable Banach space and that $f : X \to c_0^+$ is a Lipschitz embedding with constant λ_0. Then for any $u \in X$ with $\|u\| = 1$ and any infinite-dimensional subspace Y of X we have*

$$\inf_{\substack{y\in Y\\ \|y\|=1}} \|u + 2y\| \le \lambda_0.$$

Proof. The proof is almost identical to that of Proposition 3.1. It follows from Proposition 4.5 that X has property $\Pi_+(\lambda)$ for any $\lambda > \lambda_0$. We consider $B_1 = -u + 2B_X$ and $B_2 = u + 2B_X$, where B_X denotes the closed unit ball of X. Suppose $\mu > \lambda_0$ and select $\mu > \lambda > \lambda_0$. Then, for some $\nu > \mu$, we can find finitely many closed sets $(U_j, V_j)_{j=1}^N$ verifying:

$$\lambda\delta(U_j, V_j) \geq 2\nu$$

and

$$\{(x, y) \in B_1 \times B_2 : \ \|x - y\| > 2\mu\} \subset \bigcup_{j=1}^N U_j \times V_j.$$

Now let E be any subspace of X of dimension greater than N and let

$$A_j = \{e \in E : \ \|e\| = 1, \ (-u + 2e, u - 2e) \in U_j \times V_j\}.$$

We then conclude the proof as in Proposition 3.1. Assume that for any $e \in S_E$, $\|u + 2e\| > \mu$. Then $A_1 \cup \cdots \cup A_N = S_E$ and so there exists e in S_E and $k \leq N$ such that e and $-e$ belong to A_k, i.e. $-u \pm 2e \in U_k$ and $u \pm 2e \in V_k$. This implies that $\delta(U_k, V_k) \leq 2$, which is a contradiction. So, there exists $e \in S_E$ with $\|u + 2e\| \leq \mu$ and we conclude as in the proof of Proposition 3.1. □

Theorem 5.2. *Suppose* $1 \leq p < \infty$.
(i) There is a Lipschitz embedding of ℓ_p *into* c_0^+ *with constant* $(2^p + 1)^{1/p}$ *and this is best possible.*
(ii) There is a Lipschitz embedding of ℓ_p^+ *into* c_0^+ *with constant* $3^{1/p}$ *and this is best possible.*

Proof. Let us prove first that ℓ_p has $\Pi_+(c_p)$ where $c_p = (1 + 2^p)^{1/p}$. The proof is very similar to that of Theorem 3.2. Let B_1 and B_2 be balls with centers a_1, a_2 and radius $r > 0$. Suppose $\mu > c_p$ and that suppose $\mu < \nu < c_p(\mu^p - 2^p)^{1/p}$. Fix $\epsilon > 0$ such that

$$c_p\big(\mu^p r^p - 2^p(r + \epsilon)^p\big)^{1/p} - 2\epsilon c_p > \nu r.$$

We select $N \in \mathbb{N}$ so that

$$\sum_{k=N+1}^\infty |a_1(k)|^p, \sum_{k=N+1}^\infty |a_2(k)|^p < \epsilon^p.$$

Let E be the linear span of $\{e_1, \ldots, e_N\}$ where (e_j) is the canonical basis of ℓ_p. Let P the canonical projection of ℓ_p onto E, $Q = I - P$ and $R = \max(\|a_1\|, \|a_2\|) + r$. Then we partition RB_E into finitely many sets $A_1, \ldots, A_m$ with diam $A_j < \epsilon$.
Now, set $U_j = \{x \in B_1 : \ Px \in A_j\}$, $V_j = \{x \in B_2 : \ Px \in A_j\}$ and

$$\mathcal{S} = \{(j, k) : \ \exists (x, y) \in U_j \times V_k : \ \|x - y\| > \mu r\}.$$

Thus we have

$$\{(x, y) \in B_1 \times B_2 : \ \|x - y\| > \mu r\} \subset \bigcup_{(j,k)\in\mathcal{S}} U_j \times V_k.$$

It remains to estimate $\delta(U_j, V_k)$ for $(j,k) \in \mathcal{S}$. Suppose $u \in U_j, v \in V_k$ and that $x \in U_j, y \in V_k$ are such that $\|x-y\| > \mu r$. Then

$$\|u-v\| \geq \|Pu - Pv\| \geq \|Px - Py\| - 2\epsilon.$$

On the other hand

$$r \geq \|x - a_1\| \geq \|Qx\| - \epsilon$$

and

$$r \geq \|y - a_2\| \geq \|Qy\| - \epsilon.$$

Thus

$$\|Qx - Qy\| \leq 2r + 2\epsilon. \tag{5.6}$$

Now

$$\mu^p r^p < \|Px - Py\|^p + \|Qx - Qy\|^p \leq \|Px - Py\|^p + 2^p(r+\epsilon)^p.$$

Hence

$$\|Px - Py\|^p > \mu^p r^p - 2^p(r+\epsilon)^p,$$

and so

$$c_p \delta(U_j, V_k) \geq c_p\big(\mu^p r^p - 2^p(r+\epsilon)^p\big)^{1/p} - 2\epsilon c_p > \nu r.$$

Hence ℓ_p has $\Pi_+(c_p)$.

Next we show that ℓ_p^+ has property $\Pi_+(3^{1/p})$. To do this we repeat the argument above. We take $\mu > 3^{1/p}$ and suppose that $\mu < \nu < 3^{1/p}(\mu^p - 2)^{1/p}$. Choose $\epsilon > 0$ so that:

$$3^{1/p}\big(\mu^p r^p - 2(r+\epsilon)^p\big) - 2\epsilon 3^{1/p} > \nu r.$$

Next repeat the construction, but working inside the positive cone ℓ_p^+. The only difference is that (5.6) is replaced by

$$\|Qx - Qy\| \leq 2^{1/p}\max(\|Qx\|, \|Qy\|) \leq 2^{1/p}(r+\epsilon). \tag{5.7}$$

Hence

$$\|Px - Py\|^p > \mu^p r^p - 2(r+\epsilon)^p,$$

and so this time

$$3^{1/p}\delta(U_j, V_k) \geq 3^{1/p}\big(\mu^p r^p - 2(r+\epsilon)^p\big) - 2\epsilon 3^{1/p} > \nu r.$$

For the second half of the condition when $3^{1/p} \leq 2$ we note that $\varphi(x) = \|x\|$ satisfies (4.4) with $\theta = 2^{1/p} < 3^{1/p}$.

These calculations combined with Theorem 4.8 show the existence of the Lipschitz embeddings in parts (i) and (ii). Proposition 5.1 shows the constant is best possible when in (i). For (ii) let us suppose $f : \ell_p^+ \to c_0^+$ is an embedding such that

$$\|x-y\| \leq \|f(x) - f(y)\| \leq \lambda\|x-y\| \qquad x, y \in \ell_p^+$$

where $\lambda < 3^{1/p}$. Let $f(x) = (f_j(x))_{j=1}^\infty$. Let $\epsilon = (3^{1/p} - \lambda)/2$. Then there exists N such that

$$\max(f_j(e_1), f_j(0)) < \epsilon \qquad j \geq N+1.$$

Hence if $m, n > 1$

$$|f_j(e_1 + e_m) - f_j(e_n)| \leq \max(f_j(e_1+e_m), f_j(e_n)) \leq \lambda + \epsilon < 3^{1/p}, \quad j \geq N+1.$$

Now we may pass to a subsequence so that the following limits exist:

$$\lim_{k\to\infty} f_j(e_1+e_{n_k}) = \sigma_j, \quad \lim_{k\to\infty} f_j(e_{n_k}) = \tau_j, \qquad 1 \le j \le N.$$

Clearly

$$|\sigma_j - \tau_j| \le \lambda, \qquad 1 \le j \le N.$$

Now

$$\lim_{k\to\infty} |f_j(e_1+e_{n_k}) - f_j(e_{n_{k+1}})| \le \lambda \qquad 1 \le j \le N$$

and we have a contradiction since $\|e_1 + e_{n_k} - e_{n_{k+1}}\| = 3^{1/p} > \lambda$. □

6. Spaces embedding isometrically into c_0 and c_0^+.

In this final section we study isometric embeddings into c_0 and c_0^+. Note that a separable Banach space isometrically embeds into c_0 if and only if it embeds linearly and isometrically [5].

We recall that a metric space (M,d) is an ultrametric space if

$$d(x,y) \le \max(d(x,z), d(z,y)) \qquad x,y,z \in M. \tag{6.8}$$

Note that this implies

$$d(x,y) = \max(d(x,z), d(z,y)) \qquad d(x,z) \ne d(z,y). \tag{6.9}$$

Lemma 6.1. *Let (M,d) be a separable ultrametric space. Then there is a countable subset Γ of $[0,\infty)$ such that $d(x,y) \in \Gamma$ for all $x,y \in M$.*

Proof. For each fixed $x \in M$ let $\Gamma_x = \{d(x,y) : y \in M\}$. Suppose Γ_x is uncountable; then for some $\delta > 0$ the set $\Gamma_x \cap (\delta,\infty)$ is uncountable. Pick an uncountable set $(y_i)_{i\in I}$ in M so that $d(x,y_i) > \delta$ and the values of $d(x,y_i)$ are distinct for $i \in I$. Then $i \ne j$ we have $d(y_i,y_j) > \delta$ by (6.9). This contradicts separability of M.

Thus each Γ_x is countable. Let D be a countable dense subset of M and let $\Gamma = \cup_{x\in D}\Gamma_x$. If $y,z \in M$ with $y \ne z$, pick $x \in D$ with $d(x,y) < d(y,z)$. Then $d(y,z) = d(x,z) \in \Gamma$ by (6.9). □

Theorem 6.2. *Every separable ultrametric space embed isometrically into c_0^+*

Proof. Pick Γ as in Lemma 6.1. Let $(a_j)_{j=1}^\infty$ be a countable dense subset of an ultrametric space M. Let $\mathcal{D}$ be the collection of finite sequences $(r_1,\ldots,r_n)$ with $r_j \in \Gamma$ for $1 \le j \le n$. For each $(r_1,\ldots,r_n) \in \mathcal{D}$ we define a function $f_{r_1,\ldots,r_n}$ by

$$f_{r_1,\ldots,r_n}(x) = \begin{cases} \min(r_1,\ldots,r_n) & d(x,a_j) = r_j,\ 1 \le j \le n \\ 0 & \text{otherwise.} \end{cases}$$

If $x \in M$ let $d(x,a_j) = s_j$. Then $\lim_{n\to\infty} \min(s_1,\ldots,s_n) = 0$ and it follows that $f(x) = (f_{r_1,\ldots,r_n}(x))_{(r_1,\ldots,r_n)\in\mathcal{D}}$ is a map from M into $c_0^+(\mathcal{D})$.

If $x,y \in M$ and $f_{r_1,\ldots,r_n}(x) \ne f_{r_1,\ldots,r_n}(y)$ we can assume without loss of generality that $d(x,a_j) = r_j$ for $1 \le j \le n$ but that for some $1 \le k \le n$ we have $d(y,a_k) \ne r_k$. Then

$$|f_{r_1,\ldots,r_n}(x) - f_{r_1,\ldots,r_n}(y)| = \min(r_1,\ldots,r_n) \le r_k \le \max(d(x,a_k), d(y,a_k)) = d(x,y)$$

by (6.9). Thus $\|f(x) - f(y)\| \leq d(x, y)$ for $x, y \in M$.

On the other hand if $x \neq y$ there is a least k so that $d(x, a_k) \neq d(y, a_k)$. Assume $d(x, a_k) > d(y, a_k)$ and $r_j = d(x, a_j)$ for $1 \leq j \leq k$. Then $d(x, y) = r_k$. On the other hand $d(x, y) \leq r_j$ for $1 \leq j \leq k$. Hence

$$d(x, y) = r_k = |f_{r_1,\ldots,r_k}(x) - f_{r_1,\ldots,r_k}(y)|.$$

Thus f is an isometry. □

As a final example we consider an infinite branching tree $\mathcal{T}$ defined as the set of all ordered subsets (nodes) $a = (m_1, \ldots, m_k)$ (where $m_1 < m_2 < \ldots < m_k$) of $\mathbb{N}$ (including the empty set). Let $|a| = k$ be the length of a so that $|\emptyset| = 0$. If $a = (m_1, \ldots, m_k), b = (n_1, \ldots, n_l)$ are two nodes we define $a \wedge b$ to be the node $(m_1, \ldots, m_r)$ where $r \leq \min(k, l)$ is the greatest integer such that $m_j = n_j$ for $1 \leq j \leq r$. We write $a \prec b$ if $b \wedge a = a$. $\mathcal{T}$ is a graph if we define two nodes a, b to be adjacent if $||a| - |b|| = 1$ and $a \prec b$ or $b \prec a$. The natural graph metric d is thus given by

$$d(a, b) = |a| + |b| - 2|a \wedge b|.$$

Theorem 6.3. *The infinite branching tree embeds isometrically into c_0.*

Proof. For each $(a, n) \in \mathcal{T} \times \mathbb{N}$ we define

$$f_{a,n}(b) = \begin{cases} |b| - |a| & a \prec b,\ b \neq a,\ b_{|a|+1} = n \\ |a| - |b| & a \prec b,\ b \neq a,\ b_{|a|+1} > n \\ 0 & \text{otherwise.} \end{cases}$$

For fixed b we have $f_{a,n}(b) \neq 0$ only when $a \prec b$ and $n \leq b_{|a|+1}$ and this is a finite set. Hence $f(b) = (f_{a,n}(b))_{(a,n) \in \mathcal{T} \times \mathbb{N}}$ defines a map of $\mathcal{T}$ into $c_0(\mathcal{T} \times \mathbb{N})$.

Suppose $d(b, b') = 1$ and that $|b'| = |b| + 1$. Then by examining cases it is clear that $|f_{a,n}(b) - f_{a,n}(b')| \leq 1$ so that $\|f(b) - f(b')\| \leq 1$. It follows that $\|f(b) - f(b')\| \leq d(b, b')$ for arbitrary $b, b' \in \mathcal{T}$.

If $b \neq b'$ pick $a = b \wedge b'$ and assume as we may that either that $b' = a \wedge b = a$ or $b_{|a|+1} < b'_{|a|+1}$. Put $n = b_{|a|+1}$. Then

$$f_{a,n}(b) = |b| - |a|, \quad f_{a,n}(b') = |a| - |b'|$$

so that

$$|f_{a,n}(b) - f_{a,n}(b')| = d(b, b').$$

Hence f is an isometry. □

Remark. Since c_0 2-embeds into c_0^+, so does $\mathcal{T}$. It follows from the fact that $\mathcal{T}$ contains a copy of $\mathbb{Z}$, that it is again optimal.

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